Strategy for Proving Universal Sentences

1.

4. $\forall x P(x)$

We want to derive $\forall x P(x)$. Let's use $\forall$Intro.

Strategy for Proving Universal Sentences

1.		
2.	$\boxed{a}$	
$\vdots$	?	
3.	$P(a)$	
4.	$\forall x P(x)$	$\forall$Intro:2–4

For $\forall$Intro, we need a subproof that ends with a substitution instance of $\forall x P(x)$ where we replace every free occurrence of x in $P(x)$ by a *new* constant a. a has to be boxed in the assumption line.

Strategy for Proving Existential Sentences

$$\begin{array}{|l}
1. \\ \hline
\quad \vdots \\
2.\ ? \\
3.\ \exists x P(x)
\end{array}$$

We want to derive $\exists x P(x)$. For this, we can often use $\exists$Intro.

Strategy for Proving Existential Sentences

1.		
⋮	?	
2.	$P(b)$	
3.	$\exists x P(x)$	$\exists$Intro:2

For $\exists$Intro, we need a substitution instance $P(b)$ of $P(x)$. Any b will do. Be careful: Often you can only prove $P(b)$ if you're in a subproof, and b is a boxed constant in a surrounding subproof.

Strategy for Using Existential Premises

1. $\exists x P(x)$

4. Q

Suppose you want to prove some sentence Q, and you are ready to use $\exists x P(x)$. $\exists x P(x)$ might be something you've proved, or is a premise, or an assumption of a subproof).

Strategy for Using Existential Premises

$$
\begin{array}{l|l}
 & 1.\ \exists x P(x) \\
\hline
 & \quad 2.\ \boxed{c}\quad P(c) \\
 & \quad\quad \vdots \quad ? \\
 & \quad 3.\ Q \\
 & 4.\ Q
\end{array}
$$

To get Q from $\exists x P(x)$, set up a subproof where you assume P(c). c has to be *new* (in particular, it can't be in Q), and boxed. In the subproof, look for a proof of Q.

Strategy for Using Universal Premises

1. $\forall x P(x)$

To use a universal sentence which you've proved, assumed, or is one of your premises, use $\forall$Elim.

Strategy for Using Universal Premises

1.	$\forall x P(x)$	
2.	$P(a)$	$\forall$Elim:1

To do that, you can write down any substitution instance of $P(x)$, i.e., $P(a)$ where a is any constant.

Strategy for Using Universal Premises

1.	$\forall x \forall y \forall z\, P(x, y, z)$	
2.	$P(a, b, c)$	$\forall$Elim:1

If there is more than one $\forall$ (they have to be together "in a block"), you can replace all variables at once. Here we replaced x by a, y by b, and z by c.

Strategy for Using Universal Premises

1.	$\forall x \forall y \forall z\, P(x, y, z)$	
2.	$P(a, b, c)$	$\forall$Elim:1
3.	$P(c, a, c)$	$\forall$Elim:1

You don't have to keep the alphabetical order, and the constants don't all have to be distinct. E.g., line 3 comes from line 1 by replacing x by c, y by a, and z by c.

An Example

We'll now apply these strategies (and some strategies we remember from propositional proofs) to give a proof of

$$\exists x(P(x) \rightarrow Q) \rightarrow (\forall x P(x) \rightarrow Q)$$

1.

9. $\exists x(P(x) \rightarrow Q) \rightarrow (\forall x P(x) \rightarrow Q)$

The sentence we want to prove is a conditional, so we should use $\rightarrow$Intro as the last step.

1.

2. $\exists x(P(x) \rightarrow Q)$

8. $\forall x\, P(x) \rightarrow Q$

9. $\exists x(P(x) \rightarrow Q) \rightarrow (\forall x P(x) \rightarrow Q)$ $\rightarrow$Intro:2–8

We set up a subproof that starts with the antecedent and ends with the consequent. The consequent is itself a conditional, so. . .

1.		
2.	$\exists x(P(x) \rightarrow Q)$	
3.	$\forall x\, P(x)$	
7.	Q	
8.	$\forall x\, P(x) \rightarrow Q$	$\rightarrow$Intro:3–7
9.	$\exists x(P(x) \rightarrow Q) \rightarrow (\forall x P(x) \rightarrow Q)$	$\rightarrow$Intro:2–8

. . . we set up another subproof. Now the goal sentence is Q, an atomic sentence. We can't work backward anymore—let's start using assumptions, like the existential sentence on 2.

1.		
2.	$\exists x(P(x) \to Q)$	
3.	$\forall x\, P(x)$	
4.	$\boxed{a}$ $\quad P(a) \to Q$	
5.	$?$	
6.	Q	
7.	Q	$\exists$Elim:2, 4–6
8.	$\forall x\, P(x) \to Q$	$\to$Intro:3–7
9.	$\exists x(P(x) \to Q) \to (\forall x P(x) \to Q)$	$\to$Intro:2–8

To use line 2, we need a subproof that starts with a substitution instance and ends with our goal sentence, Q. So how to get Q in that subproof?

1.

2. $\exists x(P(x) \rightarrow Q)$

3. $\forall x\, P(x)$

4. $\boxed{a}$ $\quad P(a) \rightarrow Q$

5. $P(a)$ — $\forall$Elim: 3

6. Q — $\rightarrow$Elim: 4, 5

7. Q — $\exists$Elim: 2, 4–6

8. $\forall x\, P(x) \rightarrow Q$ — $\rightarrow$Intro: 3–7

9. $\exists x(P(x) \rightarrow Q) \rightarrow (\forall x P(x) \rightarrow Q)$ — $\rightarrow$Intro: 2–8

We use $\forall$Elim to get $P(a)$. This, with line 4 and $\rightarrow$Elim, gives us Q. We're done.